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From Dirac's de Sitter Equation to a Generalization of Gravitational Theory

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Dirac's de Sitter covariant spinor wave equation, expressed on the group manifold of $G = SO(3, 2)$ in terms of invariant vectors, gives rise to a modified Kaluza–Klein (KK) formalism with the Lorentz subgroup $H = SO(3, 1)$ as typical fiber and the anti-de Sitter (AdS) space as base of the principal fiber bundle on G. The resulting gauge theory of gravitation gives an incomplete description of spinning particles because spin, considered as a generalized charge, has space-time properties. It is shown how the geometrical structure has to be altered to further modify the gauge formalism to a consistent theory of spin and gravitation. The resulting gravitational field equations on the bundle include torsion, and their projection on space-time is of higher nonlinearity in the curvature.

> Ihr naht euch wieder schwankende Gestalten die früh sich einst dem trüben Blick gezeigt. Versuchich wohl euch diesmal fest zu halten? Fühlich mein Herz noch jenem Wahn geneigt?
>
> —Goethe

An early study of Sommerfeld's superb presentation of Dirac's spinor equation, expressed solely in terms of Clifford algebras without matrices, convinced the author that the Clifford algebra over complex numbers is a generalization of complex phases in wave equations (Sommerfeld, 1944). An article of Schrödinger on the Dirac equation in a gravitational field led then to the conviction that such a phase might lead to a starting phase for a gauge theory of gravitation (Schrödinger, 1932). Klein had recognized earlier that the theory of gravitation may be derived from the invariance properties of the Dirac equations. Schrödinger also directed my attention to an article of Einstein and Mayer in which quaternions over the field of complex numbers

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replace the complex Clifford numbers (Hestenes and Sobczyk, 1986). A breakthrough occurred when I became acquainted with Dirac's work on the electron wave equation in de Sitter space (Dirac, 1935). Dirac constructed a modified version of his equation with a wave operator expressed only in terms of representations of the 10 generators of the de Sitter groups. The wave operator thus becomes

$$a, b = 1 \ldots 5, \qquad \gamma^a \gamma^b M_{ab} \tag{1}$$

with $\gamma_1 \ldots \gamma_4$ the generators of the Clifford algebra $\gamma^5 = \gamma^1\gamma^2\gamma^3\gamma^4$ and $\gamma^{ab} = \gamma^a\gamma^b$,

$$M_{ab} = x_a \frac{\partial}{\partial x^b} - x_b \frac{\partial}{\partial x^a} \tag{2}$$

The de Sitter and anti-de Sitter spaces of constant curvature are imbedded as four-dimensional spheres into five-dimensional pseudo-Euclidean space with the metrics

$$\eta_{aa} \equiv (1, 1, 1, \pm 1, -1), \qquad \eta_{ab} \equiv 0 \quad (a \neq b), \qquad \eta_{ab} x^a x^b = \pm r^2 \tag{3}$$

Dirac showed that the operators $M_{ab}M^{ab}$ and $\gamma^{ab}M_{ab}$ in neighborhoods of the point $x^5 \approx r$, x^k small ($k = 1 \ldots 4$), are approximations of the Klein–Gordon and Dirac wave operators (apart from factors r^2 and $r\gamma^5$),

$$i, k = 1 \ldots 4, \qquad -r^2 \eta^{ik} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^k} \qquad \text{and} \qquad -r\gamma^5\gamma^k \frac{\partial}{\partial x^k} \tag{4}$$

Dirac then constructed the analogue of the conventional expressions in the modified theory.

I took up the subject in the 1970s, presenting results occasionally at meetings in honor of Dirac (Halpern, 1975, 1982, 1984, 1994, 1996, 1998). The theory was soon formulated on the manifold of the simple anti-de Sitter group $G = SO(3, 2)$ or its universal covering group. Such formulations had become a prevailing trend, mainly for the Poincaré group (Bopp and Haag, 1950; Neeman and Regge, 1978; Gusiew and Keller, 1997). The Poincaré group offers the reassuring property of making particle rest masses Casimir invariants. This is not true for our simple group G. The volume of phase space of zero-rest-mass radiation exceeds by far the corresponding volume available to massive matter, so that transitions from the latter to the former without protective conservation laws for rest masses pose a threat. Transitions of this kind may occur, however, only within cosmological time spans. Approximate conservation laws have proven after all most beneficial to the development of science.

The anti-de Sitter space is obtained as a factor space G/H with $G = SO(3, 2)$ and $H = SO(3, 1)$ the Lorentz subgroup. Simple groups have no invariant subgroups, so that the left and right cosets do not coincide. We choose the space of the left cosets.

A principal fiber bundle $P(G, H, \pi, G/H)$ with typical fiber H, natural projection π: $G \to G/H = B$, is formed (Nomizu, 1956). Because it is trivial, one can introduce noncanonical coordinates on G: x^r, $r = 1 \ldots 10$, for which the coordinates x^h ($h = 1 \ldots 4$) serve not only to label points of G, but also their projection points on B. The remaining 6 coordinates x^m, $m = 5 \ldots 10$, label the points of the fiber over each point of B.

We can construct left-invariant vector fields A_R, $R = 1 \ldots 10$, such that the last 6, A_M ($M = 5 \ldots 10$), lie on the fibers,

$$R, S, T = 1 \ldots 10, \qquad [A_R, A_S] = c_{RS}{}^T A_T \tag{5}$$

The remaining 4 vector fields A_K ($K = 1 \ldots 4$) can be chosen perpendicular to the fibers, and all 10 perpendicular to each other w.r.t. the Cartan–Killing metric γ,

$$\gamma_{RS} = \mathrm{Tr}(\mathscr{A}d(A_R), \mathscr{A}d(A_S)) = c_{RU}{}^V c_{SV}{}^U \tag{6}$$

Expressed in terms of the double-index notation used by Dirac, the structure constants have the value

$$c_{[a,b][c,d]}{}^{[e,f]} = \eta_{ac}\delta_b^e\delta_d^f \tag{7}$$

with the necessary changes of sign with permutation of indices within a bracket. The simple group also implies antisymmetry to exchange of any pair of double indices: $c_{RS}{}^T = -c_{RT}{}^S$, etc.

A connection is defined by the metric γ with horizontal vectors perpendicular to the fiber. Horizontal vectors are henceforth denoted by indices $A \ldots K$ running from 1 to 4, and vertical vectors by indices $L \ldots Q$ that run from 5 to 10. Letters $R \ldots Z$ denote any indices $1 \ldots 10$. The corresponding summation convention is applied without further warning. Thus ${}^{B}_{B}$ stands for $\sum_{B=1}^{4}$, ${}^{M}_{M} = \sum_{M=5}^{10}$, ${}^{S}_{S} = \sum_{S=1}^{10}$, these rules applying also to coordinate indices (lowercase letters). The forms dual to the left-invariant vectors are denoted by A^R. Right-invariant vectors and forms are denoted by a bar: $\overline{A}_R$, $\overline{A}^S$,

$$[\overline{A}_R, \overline{A}_S] = -c_{RS}{}^T\overline{A}_T, \qquad [A_R, \overline{A}_S] = 0 \tag{8}$$

The natural projection π defines the projection of the horizontal part of a vector on the bundle

$$\pi' A_E(x^r) = a(x^e); \qquad \pi' A_M = 0 \tag{9}$$

The commutation relations (5) imply that $\pi' A_E$ depends on the point of the fiber at which A_E is chosen, whereas $\pi'\overline{A}_E$ according to the second equation of (8) does not depend on it. The bundle of orthonormal frames is equivalent to the projections $\pi' A_E$ for a left-invariant vector field from any point of the fiber for every point of B. It is isomorphic to P. The metric γ on G determines thus a metric $\pi'\gamma = g$ on B.

Physicists would tend to call the vectors A_M in the present case "internal" rather than vertical. They are obviously acting on the internal degrees of freedom of spin. The right-invariant $\overline{A}_M$ act on space-time as well. The horizontal A_E act on space-time and have mainly coordinate components A_E^e. Their dual forms have exclusively components A_k^E ($A_m^E = 0$). Coordinate components A_E^m and $A_k^N \neq 0$ are a result of space-time curvature. Always A_M^k, A_n^E are zero.

The composition of $\overline{A}_R$ of horizontal and vertical vectors depends on the arbitrary choice of the origin on G,

$$\overline{A}_R(x) = Ad_{\alpha(x)}(A_R(x)) \tag{10}$$

with $\alpha(x)$ the group element on the point of G with coordinates x. Vertical components of $\overline{A}_E$ thus vanish at the origin of G and increase with horizontal distance from it.

The wave operator equation (1) that Dirac formed may be interpreted as a representation of the Casimir operator $\gamma^{RS} A_R A_S$ of G.

I was intrigued by the distinction of horizontal and vertical in the present formulation. It appears like a model of orbital and inner angular momentum. We observe that Dirac's wave operator uses generators of G of which one kind acts only on spin variables and the other kind only on space-time. The latter operators on the group manifold G must be the space-time component of a right-invariant generator, whereas the other kind corresponds to a left-invariant generator. The present formulation allows one to construct a corresponding master equation for any spin,

$$\begin{aligned} D(x) &= \gamma^{RS}\,\mathrm{Ver}(\overline{A}_R)\mathrm{Hor}(\overline{A}_S) \\ &= \gamma^{RS}\overline{A}_R^u A_u^M A_M^m \overline{A}_S^v A_v^E A_B^t \frac{\partial}{\partial x^m}\frac{\partial}{\partial x^t} \end{aligned} \tag{11}$$

An operator D acts on the wave function $\psi(x^r)$, which is a functional realization equivalent to a linear representation (Bopp and Haag, 1950). In case of vanishing rest mass, $D\psi(x) = 0$. The only Clifford algebra which occurs here explicitly is that of the complex numbers in the wave function. The burden rests on the functional structure of the wave function to provide the required functional realizations.

Dirac's wave operator is too special to be generalizable uniquely to a Casimir operator. I just mention a different generalization which becomes the sum of the different adjoint transformations of the 10 degrees of freedom of G. A particle of vanishing rest mass has to fulfill, accordingly,

$$\sum_{u=1}^{10} \mathscr{A}d_{A_U}\psi = 0 \tag{12}$$

Here $\mathscr{A}d_{A_U} = (R'_{A_U})^{-1}L'_{A_U}$ with R and L right and left group transformations. ψ must then be the functional realization of an element of the Lie algebra, so that simply,

$$\sum_{U=1}^{10} c_{US}{}^{T}\psi_T = 0$$

In suitable domains, this may agree with Dirac's wave operator, although the operator of Eq. (12) contains no projection operators like that of Eq. (11) and thus always contains a Klein–Gordon operator bilinear in the derivatives which vanishes for zero rest mass. We shall not deal further with this odd interpretation.

The Cartan–Killing metric of a simple group fulfills the relation with R_{uv} the Riccí tensor on G (Eisenhart, 1933),

$$R_{uv} = \tfrac{1}{4}\gamma_{uv}$$

The author interpreted this early as equivalent to γ fulfilling the Einstein vacuum equation with a cosmological term of unity,

$$R_{uv} - \tfrac{1}{2}\gamma_{uv}R + \gamma_{uv} = 0 \tag{13}$$

A_R and $\overline{A}_R$ are each Killing vector fields on G and their orbits are geodesics.

The projection by π' of Eq. (13) onto B results in Einstein's equations on B with a cosmological term corresponding to the "radius" of the universe of the anti-de Sitter group. A unit of length is thereby established on B.

The projection of orbits of the group generated by a horizontal A_E results in geodesics of the metric g on B. Projection of orbits of vectors with horizontal as well as suitable vertical components onto B results in lines which correspond to the motion of spinning test particles. Tangent vectors of particle motion expressed in terms of A_R must have a timelike component of A_E and the horizontal and vertical parts must commute.

The geometrical relations of the group manifold exposed are the ingredients for the construction of a non-Abelian Kałuża–Klein-type gauge theory (KK theory).

I developed doubts about the unlimited validity of the Einstein–Hilbert equations of general relativity because they imply inevitably the collapse of matter to a point and cannot account for Schrödinger's discovery of pair formation in gravitational fields.

I saw at least one of these shortcomings in the limitations to which a mathematical structure based on the idealization of axioms necessarily is subject to in describing objects in nature.

Galileo's principle of inertia makes use of such abstractions as straight lines in Euclidean space. This principle is at the foundation of all physical theory, but we know that in the large, space is not flat and in the small, particles have spin.

A modernized, relativistic version of the principle could be: A particle (spinning or structureless) moves along an orbit of the anti-de Sitter group in the anti-de Sitter universe unless .... A corresponding version of the principle of equivalence, locally adapted to anti-de Sitter space, is easily constructed (Halpern, 1982, 1984). The present Kałuża–Klein formalism suggests one generalize the Cartan–Killing metric to more general metrics γ which are solutions of the Einstein equations (13) and have the same metric on the fibers as before. Also, six Killing vector fields A_M should remain intact. The new metric γ determines, however, four different horizontal vector fields A_E. They are perpendicular to the A_M with the new metric γ. There are in general no right-invariant vectors. The commutation relations of the A_M with the A_E and with each other should remain unaltered as in Eq. (5). Only we have now

$$[A_E, A_F] = \mathscr{C}_{EF}{}^R A_R \tag{14}$$

where the $\mathscr{C}$ are functions of the points on G. In particular, $\mathscr{C}_{EF}{}^M$ are the components of the Lie algebra-valued curvature two-form and $\mathscr{C}_{EF}{}^H$ the torsion two-form, which vanishes for the Cartan–Killing metric and for many other solutions γ of Eq. (13). The Jacobi relations must be fulfilled at every point also with the $\mathscr{C}_{EF}(x)^R$ replacing $c_{EF}{}^R$; this guarantees the correct symmetry properties of the related tensorial expressions on the base B. The new solutions retain the topology of G.

The field equations are Einstein's equations in 10 dimensions with cosmological member. Solutions must respect the six Killing vector fields A_M and the topology of G.

Expressing the Christoffel connection in an orthonormal frame and considering the physical interpretation of Eq. (14) as curvature and torsion two-forms, we easily project the equations onto the base and express them there in curvature and contortion tensors. The most general admissible connections resulting on the base are those of the Riemann–Cartan geometry of the form,

$$\begin{aligned} \Gamma^i_{jk} &= \left\{ {}_{j\,k}^{\,i} \right\} - K_{jk}{}^i \\ K_{jk}{}^i &= -K_j{}^i{}_k = T_{kj}{}^i + T_k{}^i{}_j + T_j{}^i{}_k \\ T_{jk}{}^i &= \tfrac{1}{2}(\Gamma^i_{jk} - \Gamma^i_{kj}) \end{aligned} \tag{15}$$

The geometry on the base is a metric one (vanishing covariant derivatives) of metric tensors. For an introduction to this geometry in a quite different context see Hehl *et al.* (1976). The field equations on the base are

$$B_{ik} - \tfrac{1}{2} g_{ik}(B + 1) = \tfrac{3}{2}\kappa[H_{ijhl}H_k^{\ jhl} - \tfrac{1}{4} g_{ik} H_{abcd} H^{abcd}] = 0$$
$$H^k_{\ jhl;k} = 0 \tag{16}$$

B_{ik} and B are the Ricci tensor and invariant on the base and H is the curvature tensor, which, different from B, may include contortion (B is not to be confounded with the symbol for the base manifold). The constant κ should be 1 in our notation. We insert it as a generalization which can even depend on space-time similar as Einstein did in his work on Kaluza–Klein theory. The equations can have solutions which violate the principle of equivalence and prevent collapse to a point (Halpern, 1994, 1996). The solutions for vanishing contortion include all solutions of Einstein's equations with cosmological term.

Although we have constructed a theory of the Kaluza–Klein type, it becomes clear that it can be at best an approximation. The motion of a spinning particle shows the correct interaction of spin and curvature, but not the spin precession. The formalism has to be generalized to the case of a charge with space-time components like spin. This can be achieved without altering the metric properties and commutation relations by extending the action of the Lie algebra of the connection one-form also to the fibers. Formally, the introduction of a contortion term for the components of the linear connection on G on the fibers, which is equivalent to mixed vertical–horizontal components, acting on horizontal vectors, achieves the goal (Halpern, 1998). Notice that this is contortion on G, to be distinguished from that on B mentioned previously. Seen from the point of view of Kaluza–Klein theories, the Einstein equations (13) in 10 dimensions have to be generalized to include these terms with the contortion tensor. Projected on the base, this results there in scalar fields depending on the curvature on B. The nonlinearity of these complete equations in the curvature is of course greater than that of Eqs. (13) and (16). This completes the theory doing justice to a generalization of the principle of inertia. The generalization required to include spin precession was rather obvious, but the modification of the geometry to generalize this formalism without disturbing it was not readily found.

The term to include spin precession could easily be inferred from the general relativistic version of the Dirac equation where the spin connection acts on the spin of the wave function. The incorporation of this effect into the geometry to generalize the Kaluza–Klein formalism posed, however, considerable problems. The Kaluza–Klein formalism is introduced on the anti-de Sitter manifold; the gauge group is $H = SO(3, 1)$. The second set of the coupled equations (16) were already suggested independently by C. Kilmister and by C. N. Yang. The equations' nonlinearity in the curvature is increased by the contortion on the group manifold.

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